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Operator Formulation of Classical Mechanics

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By making use of the Weyl–Wigner–Groenewold–Moyal association rules, a commutative product and a new quantum bracket are constructed in the space of operators $\mathcal{F}(\mathcal{H})$. In this way, an isomorphism between the Lie algebra of classical observables (with Poisson bracket) and the Lie algebra of quantum observables with this new bracket is established. By these observations, a formulation of classical mechanics in $\mathcal{F}(\mathcal{H})$ is obtained and is shown to be the $\hbar \rightarrow 0$ limit of the Heisenberg-picture formulation of quantum mechanics.

1. INTRODUCTION

In this paper we answer, in a most general setting, two related questions: (1) What is the analogue of the multiplicative structure of classical observables (functions defined on classical phase space) in the quantum formalism? (2) "What is the image of the Poisson bracket (PB) of functions in the space of operators" (of quantum observables)? These questions, in one way or another, have been in the minds of many physicists since the very beginning of quantum mechanics [1, 2]; to the best of my knowledge, they remain unanswered. The second question was explicitly stated, as quoted above, in the second of the two seminal papers in ref. 3, where the figure of an empty box was used for the image of the PB.

Throughout this paper we assume that the phase space is $\mathbf{R}^{2d}$, d integer, and that the Hilbert space $\mathcal{H}$ on which the operators act is the space of square-integrable functions $L^2(\mathbf{R}^d)$. We consider the space of operators $\mathcal{F}(\mathcal{H})$ to be the universal enveloping algebra of the Heisenberg–Weyl (HW) algebra generated by $[\hat{q}, \hat{p}\,] = i\hbar\hat{I}$, where $\hbar = h/2\pi$, $\hat{I}$, $\hat{q}$, and $\hat{p}$ are Planck's constant, the identity operator, and the Hermitian position and momentum operators,

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respectively. Here and henceforth operators and functions of operators acting in $\mathscr{H}$ are denoted by a caret. The linear transformations of $\mathscr{F}(\mathscr{H})$ will be referred to as superoperators and they will be represented by a caret atop boldface letters.

In the well-known canonical quantization [1, 2], for the analogue of multiplication, the "product $\rightarrow$ anticommutator" rule works well up to cubic polynomials. For the analogue of PB, the "PB $\rightarrow (i\hbar)^{-1}$ commutator ([,])" rule works well up to quadratic polynomials of position and momentum variables, and up to observables which are affine functions of the position or of the momentum. According to the Groenewold–van Hove theorem, they lead to inconsistencies for quartic and cubic polynomials, respectively. Similar difficulties arise for other phase spaces which have topology different from $\mathbf{R}^{2d}$. For a more extensive and technical discussion of this topic see refs. 2 and 4 and references therein.

The questions posed above underline the fundamental differences between classical mechanics and quantum mechanics. We will answer them by making use of the Weyl–Wigner–Groenewold–Moyal (WWGM) quantization scheme (for recent reviews see refs. 5 and 6). The WWGM quantization enables us to carry the quantum theory to a phase space and give it an autonomous structure [3] with its own "genvalue" equations [7], (quasi)probability distributions [5], and spectral resolution. Quantum information encoded in the noncommutative product of the quantum observables is transferred via the WWGM association to classical phase space and stored in the noncommutative $*$-product [see Eq. (16) below] of classical observables. In this way, to the product of operators there corresponds the $*$-product of functions and to the commutator of operators there corresponds the Moyal bracket (MB) of functions. The resulting theory is also referred to as deformation quantization, or the phase-space formulation of quantum mechanics. On the other hand, this quantization scheme also enables us to carry classical mechanics on a phase space to a Hilbert space. This paper concentrates on the latter aspect of this quantization, although almost all the literature deals with the former. More concretely, we seek what corresponds to the commutative product of functions and to their PB in the WWGM quantization. Our answers to these questions will lead us to a formulation of classical mechanics in $\mathscr{F}(\mathscr{H})$. Moreover, the rules of association between functions and operators that emerge from this work are free from the difficulties stated by the Groenewold–van Hove theorem.

For the purposes of this paper we mainly consider systems with one degree of freedom ($d = 1$) and the corresponding phase spaces in real coordinates. Generalizing our results to systems with a finite or a denumerably infinite number of degrees of freedom and to phase spaces with complex coordinates is straightforward. Such a generalization of one of the main results of this paper is given at the end of Section 4. We use the derivative-based approach developed in ref. 9. This is different from the conventional integral-based approach, but can be considered a Liouville-space formalism [10].

The organization of the paper is as follows. In Section 2 we define some important superoperators which are commutative functions of their arguments. The importance of these superoperators is made manifest in Section 3, which includes a brief review of some fundamental ideas of the WWGM quantization, and our answer for the first question. As the second main result of this paper, a new quantum bracket is derived in Section 4. In Section 5 we give some general applications by using this new bracket. We conclude with a brief summary and discussion of results.

2. LIOUVILLIAN SUPEROPERATORS

In terms of HW algebra and a complex parameter $s \in \mathbf{C}$ we define the superoperator

$$\hat{\mathbf{O}}_{nm}^{(s)} = 2^{-(n+m)}\hat{\mathbf{T}}_{[\hat{q}](s)}^{n}\hat{\mathbf{T}}_{[\hat{p}](-s)}^{m} \tag{1}$$

where $\hat{\mathbf{L}}_{\hat{A}}$ and $\hat{\mathbf{R}}_{\hat{A}}$ are, respectively, multiplication from left and from right by $\hat{A}$,

$$\hat{\mathbf{T}}_{[\hat{A}](s)} = (1+s)\hat{\mathbf{L}}_{\hat{A}} + (1-s)\hat{\mathbf{R}}_{\hat{A}} \tag{2}$$

Note that for an arbitrary operator $\hat{F}$

$$[\hat{\mathbf{T}}_{[\hat{q}](s)}, \hat{\mathbf{T}}_{[\hat{p}](-s)}]\hat{F} = 0 \tag{3}$$

The actions of $\hat{\mathbf{O}}_{nm}^{(s)}$ on the unit operator $\hat{I}$ and, under the trace sign, on an arbitrary operator $\hat{F}$ are as follows:

$$\hat{\mathbf{O}}_{nm}^{(s)}(\hat{I}) = \hat{t}_{nm}^{(s)} \tag{4}$$

$$Tr[\hat{\mathbf{O}}_{nm}^{(s)}(\hat{F})] = Tr[\hat{t}_{nm}^{(-s)}\hat{F}] \tag{5}$$

where

$$\begin{aligned}\hat{t}_{nm}^{(s)} &= 2^{-(n+m)}\hat{\mathbf{T}}_{[\hat{q}](s)}^{n}\hat{\mathbf{T}}_{[\hat{p}](-s)}^{m}\hat{I}\\ &= 2^{-(n+m)}\hat{\mathbf{T}}_{[\hat{p}](-s)}^{m}\hat{\mathbf{T}}_{[\hat{q}](s)}^{n}\hat{I} \qquad (6)\\ &= 2^{-n}\sum_{j=0}^{n}\binom{n}{j}(1+s)^{j}(1-s)^{n-j}\hat{q}^{j}\hat{p}^{m}\hat{q}^{n-j}\\ &= 2^{-m}\sum_{k=0}^{m}\binom{m}{k}(1-s)^{k}(1+s)^{m-k}\hat{p}^{k}\hat{q}^{n}\hat{p}^{m-k} \qquad (7)\end{aligned}$$

is the s-ordered product of a term containing n factors of $\hat{q}$ and m factors of $\hat{p}$. In obtaining these expressions, we note that ordering parameters $-s$ and s in the last factors of the first two lines of the above expressions do not contribute to the results since $\hat{\mathbf{T}}^{m}_{[\hat{A}](\pm s)}\hat{I} = 2^m\hat{A}^m$. Moreover, in the last two lines we made use of the binomial formula

$$\hat{\mathbf{T}}^{n}_{[\hat{A}](s)} = \sum_{j=0}^{n}\binom{n}{j}(1+s)^j(1-s)^{n-j}\hat{\mathbf{L}}^{j}_{\hat{A}}\hat{\mathbf{R}}^{n-j}_{\hat{A}} \tag{8}$$

From Eqs. (6) and (7) we have, for $s = \pm 1$,

$$\hat{t}^{(1)}_{nm} = \hat{\mathbf{L}}^{n}_{\hat{q}}\hat{\mathbf{R}}^{m}_{\hat{p}}\hat{I} = \hat{q}^n\hat{p}^m, \qquad \hat{t}^{(-1)}_{nm} = \hat{\mathbf{L}}^{m}_{\hat{p}}\hat{\mathbf{R}}^{n}_{\hat{q}}\hat{I} = \hat{p}^m\hat{q}^n \tag{9}$$

and for $s = 0$,

$$\hat{t}^{(0)}_{nm} = 2^{-n}\sum_{j=0}^{n}\binom{n}{j}\hat{q}^j\hat{p}^m\hat{q}^{n-j} = 2^{-m}\sum_{k=0}^{m}\binom{m}{k}\hat{p}^k\hat{q}^n\hat{p}^{m-k} \tag{10}$$

While relations (9) exhibit the standard ($s = 1$) and antistandard ($s = -1$) rule of ordering, those corresponding to $s = 0$ are two well-known expressions of the Weyl, or symmetrically ordered products. In fact, the usual expression for the Weyl ordered form of $\hat{t}^{(0)}_{nm}$ is a totally symmetrized form containing n factors of $\hat{q}$, and m factors of $\hat{p}$, normalized by dividing by the number of terms in the symmetrized expression. Here we give an example:

$$\hat{t}^{(0)}_{12} = \frac{1}{2}(\hat{q}\hat{p}^2 + \hat{p}^2\hat{q}) = \frac{1}{4}(\hat{q}\hat{p}^2 + 2\hat{p}\hat{q}\hat{p} + \hat{p}^2\hat{q}) = \frac{1}{3}(\hat{q}\hat{p}^2 + \hat{p}\hat{q}\hat{p} + \hat{p}^2\hat{q})$$

As a simple result of the approach followed here, explicit expressions for many forms of the s-ordered products and their equivalences, without using the usual commutation relations, naturally arise by noting only the relation (3). From (7) it easily follows that $[\hat{t}^{(s)}_{nm}]^\dagger = \hat{t}^{(-\bar{s})}_{nm}$, that is, for general n, m integers, $\hat{t}^{(s)}_{nm}$ are Hermitian if and only if $\bar{s} = -s$ ($\bar{s}$ denotes the complex conjugation of s). In particular, the Weyl ordered products $\hat{t}^{(0)}_{nm}$ are Hermitian.

Because of Eqs. (4) and (5), we call $\hat{\mathbf{O}}^{(s)}_{nm}$ the ordering superoperator. Making use of (1) and (5), we obtain the following relations for its repeated action:

$$\hat{t}^{(s)}_{nm} = \hat{\mathbf{O}}^{(s)}_{nm}(\hat{I}) = \hat{\mathbf{O}}^{(s)}_{n_1m_1}(\hat{\mathbf{O}}^{(s)}_{n_2m_2}(\hat{I})) = \hat{\mathbf{O}}^{(s)}_{n_1m_1}(\hat{\mathbf{t}}^{(s)}_{n_2m_2}) \tag{11}$$

$$\begin{aligned} Tr[\hat{\mathbf{O}}^{(s)}_{nm}(\hat{F})] &= Tr[\hat{\mathbf{O}}^{(s)}_{n_1m_1}(\hat{\mathbf{O}}^{(s)}_{n_2m_2}(\hat{F}))] \\ &= Tr[\hat{t}^{(-s)}_{n_1m_1}(\hat{\mathbf{O}}^{(s)}_{n_2m_2}(\hat{F}))] \\ &= Tr[\hat{F}\hat{\mathbf{O}}^{(-s)}_{n_2m_2}(\hat{t}^{(-s)}_{n_1m_1})] \end{aligned} \tag{12}$$

where $n = n_1 + n_2$, $m = m_1 + m_2$. Here is an example of the relation (11):

$$\hat{t}^{(s)}_{n+1,m+1} = \frac{1}{4}\hat{\mathbf{T}}_{[\hat{q}](s)}\hat{\mathbf{T}}_{[\hat{p}](-s)}(\hat{t}^{(s)}_{nm})$$
$$= \frac{1}{4}\{(1 - s^2)[\hat{q}\hat{p}\hat{t}^{(s)}_{nm} + \hat{t}^{(s)}_{nm}\hat{p}\hat{q}] + (1 + s)^2\hat{q}\hat{t}^{(s)}_{nm}\hat{p}$$
$$+ (1 - s)^2\hat{p}\hat{t}^{(s)}_{nm}\hat{q}\} \tag{13}$$

Finally, in this section, with a phase-space function expandable as power series in p and q

$$f(q, p) = \sum_{n,m} c_{nm}q^n p^m \tag{14}$$

we associate a Liouvillian superoperator

$$\hat{\mathbf{f}}^{(s)} = \hat{\mathbf{f}}\left(\frac{1}{2}\hat{\mathbf{T}}_{[\hat{q}](s)}, \frac{1}{2}\hat{\mathbf{T}}_{[\hat{p}](-s)}\right) = \sum_{n,m} c_{nm}\hat{\mathbf{O}}^{(s)}_{nm} \tag{15}$$

Note that, like $f(q, p)$, $\hat{\mathbf{f}}^{(s)}$ is also a commutative function of its arguments, and in this sense the Liouvillian superoperators defined here mimic the fundamental property of the corresponding phase-space functions in $\mathcal{F}(\mathcal{H})$. $\hat{\mathbf{O}}^{(s)}_{nm}$ is the Liouvillian superoperator corresponding to the monomial $q^n p^m$. For brevity, we write $\hat{\mathbf{f}}^{(s)}$ and $\hat{\mathbf{O}}^{(s)}_{nm}$ without denoting their arguments.

3. WWGM QUANTIZATION

Let us denote by $N = C^\infty(M)$ the vector space of functions defined over a phase space M. While with the usual pointwise product N becomes a commutative and associative algebra, $\mathcal{F}(\mathcal{H})$ becomes a noncommutative, but associative algebra with respect to usual operator product. A noncommutative, but associative algebra structure on N can be implemented by the $*$-product; $*_{(-s)}$: $N \times N \to N$, explicitly given by[3]

$$*_{(-s)} = \exp\frac{1}{2}i\hbar[(1 - s)\,\partial^L_p\,\partial^R_q - (1 + s)\,\partial^L_q\,\partial^R_p] \tag{16}$$

Here we take $(q, p) \in \mathbf{R}^2 = M$ and use the convention that ∂^L and ∂^R are acting on the left (L) and on the right (R), respectively. Thus two different Lie algebra structure can be defined on N; with respect to PB, $\{\ ,\ \}_{\mathrm{PB}}$: $N \times N \to N$, defined by

[3] Denoting by $N[[\nu]]$ the space of formal power series in the parameter ν with coefficients in N, both the star product and MB can be considered as $N[[\nu]] \times N[[\nu]] \to N[[\nu]]$. The so-called deformation parameter ν in physical applications corresponds to $i\hbar/2$.

$$\{f, g\}_{PB} = \partial_p f\, \partial_q g - \partial_q f\, \partial_p g \tag{17}$$

(henceforth the notation $\partial_x \equiv \partial/\partial x$ will be used), and with respect to s-MB, defined by

$$\{f_1(q, p), f_2(q, p)\}_{MB}^{(-s)} \equiv f_1(q, p) *_{(-s)} f_2(q, p) - f_2(q, p) *_{(-s)} f_1(q, p) \tag{18}$$

where $f_1, f_2 \in N$. Let us denote these two Lie algebras by N_{PB} and N_{MB}, where PB and MB refer to the respective brackets. The relations (16) and (18) unify the different expressions for the star product and Moyal brackets which appear in the literature and generalize them for an arbitrary s-ordering [9].

Despite these two different Lie algebra structures there is only one in $\mathcal{F}(\mathcal{H})$ defined with respect to the usual Lie bracket [,], which we denote by $\mathcal{F}_{LB}$. This is (anti-)homomorphic to Lie algebra N_{MB}: $\{q^n p^m, q^k p^l\}_{MB}^{(-s)} \to -[\hat{t}_{nm}^{(s)}, \hat{t}_{kl}^{(s)}]$ [see Eq. (61) of the second paper of ref. 9]. In the next section we will obtain a new quantum bracket which, quite in parallel with the Lie algebra structures in N, enables us to define a new Lie algebra structure in $\mathcal{F}(\mathcal{H})$. Before doing that, we recall some fundamental relations of the WWGM quantization.

The above mentioned (anti-)homomorphism between N_{MB} and $\mathcal{F}_{LB}$ is established via the WWGM quantization rule symbolically defined by the linear and invertible map $\mathcal{M}_s$: $N \to \mathcal{F}(\mathcal{H})$ with inverse $\mathcal{M}_s^{-1}$: $\mathcal{F}(\mathcal{H}) \to N$, such that $\mathcal{M}_s\mathcal{M}_s^{-1}$ and $\mathcal{M}_s^{-1}\mathcal{M}_s$ are identity transformations on $\mathcal{F}(\mathcal{H})$ and N, respectively. Explicitly, we write $\mathcal{M}_s(f) = \hat{F}^{(s)}$ and $\mathcal{M}_s^{-1}(\hat{F}^{(s)}) = f$, where[4]

$$\hat{F}^{(s)}(\hat{q}, \hat{p}) = h^{-1} \int\int f(q, p)\hat{\Delta}_{qp}(s)\, dq\, dp; \qquad f(q, p) = Tr[\hat{F}^{(s)}\hat{\Delta}_{qp}(-s)] \tag{19}$$

(all the integrals are from $-\infty$ to ∞). The first relation is an expansion of an operator in a complete continuous operator basis

$$\hat{\Delta}_{qp}(s) = (\hbar/2\pi) \int\int e^{-i(\xi q+\eta p)}\hat{D}(s)\, d\xi\, d\eta \tag{20}$$

obeying the relations

[4] In this paper we want to map a specified structure defined on N to $\mathcal{F}(\mathcal{H})$. But, for different mapping rules different operators correspond to the same function. To distinguish them we label them by a superscript (s). Conversely, when the main goal is to map a specified structure defined in $\mathcal{F}(\mathcal{H})$ to N, different functions corresponding to the same operator are to be distinguished.

$$\int\int \hat{\Delta}_{qp}(s)\, dq\, dp = h, \qquad Tr[\hat{\Delta}_{qp}(s)] = 1 \tag{21}$$

Here $\hat{D}(s) = \exp(-i\hbar s\xi\eta/2)\exp i(\xi\hat{q} + \eta\hat{p})$ is the s-parametrized displacement operator. The basis operators $\hat{\Delta}_{qp}$ are known as the Grossmann–Royer displaced parity operators [11] for $s = 0$ and as the Kirkwood bases for $s = \pm 1$. Since they form complete operator bases, in the sense that any operator obeying certain conditions can be expanded in terms of them as in the first relation given by (19), they provide a unified approach to different quantization rules [8, 12]. For special values $s = 1, 0, -1$ these are known, respectively, as the standard, the Wigner–Weyl, and the antistandard rules of associations [5, 12].

The second relation in Eq. (19) easily follows by multiplying both sides of the first relation by $\hat{\Delta}_{q'p'}(-s)$ and making use of the relation

$$Tr[\hat{\Delta}_{qp}(s)\hat{\Delta}_{q'p'}(-s)] = h\delta(q - q')\delta(p - p') \tag{22}$$

Among other nice properties of the $\hat{\Delta}(s)$ basis, we quote the so-called differential properties:

$$\partial_q\, \hat{\Delta}_{qp}(s) = -\frac{i}{\hbar}[\hat{p}, \hat{\Delta}_{qp}(s)], \qquad \partial_p \hat{\Delta}_{qp}(s) = \frac{i}{\hbar}[\hat{q}, \hat{\Delta}_{qp}(s)] \tag{23}$$

$$q\hat{\Delta}_{qp}(s) = \frac{1}{2}\hat{\mathbf{T}}_{[\hat{\mathbf{q}}](s)}\hat{\Delta}_{qp}(s), \qquad p\hat{\Delta}_{qp}(s) = \frac{1}{2}\hat{\mathbf{T}}_{[\hat{\mathbf{p}}](-s)}\hat{\Delta}_{qp}(s) \tag{24}$$

These last relations can be generalized as

$$q^n p^m \hat{\Delta}_{qp}(s) = \hat{\mathbf{O}}^{(s)}_{nm}(\hat{\Delta}_{qp}(s)) \tag{25}$$

As an illustration, taking the traces of both sides, we have $q^n p^m = Tr[\hat{t}^{(s)}_{nm}\hat{\Delta}_{qp}(-s)]$, which shows that $\mathcal{M}_s(q^n p^m) = \hat{t}^{(s)}_{nm}$, or $\mathcal{M}_s^{-1}(\hat{t}^{(s)}_{nm}) = q^n p^m$. More generally, for a function accepting power series expansion as in (14), we see that the corresponding operator in s-association given by (19) is obtained by simply replacing $q^n p^m$ by $\hat{t}^{(s)}_{nm}$. For these kinds of functions a generalization of (25) is

$$f(q, p)\hat{\Delta}_{qp}(s) = \hat{\mathbf{f}}^{(s)}(\hat{\Delta}_{qp}(s)) \tag{26}$$

Now by multiplying both sides of this relation by another function $g(q, p)$ we have

$$g(q, p) f(q, p)\hat{\Delta}_{qp}(s) = \hat{\mathbf{f}}^{(s)}[\hat{\mathbf{g}}^{(s)}(\hat{\Delta}_{qp}(s))] = \hat{\mathbf{g}}^{(s)}[\hat{\mathbf{f}}^{(s)}(\hat{\Delta}_{qp}(s))]$$

By taking the integral and trace of all sides and making use of the relations (21), we arrive at

$$h^{-1} \int\int g(q,p) f(q,p) \hat{\Delta}_{qp}(s)\, dq\, dp$$
$$= \hat{\mathbf{f}}^{(s)}[\hat{\mathbf{g}}^{(s)}(\hat{I})] = \hat{\mathbf{g}}^{(s)}[\hat{\mathbf{f}}^{(s)}(\hat{I})] = \hat{\mathbf{f}}^{(s)}(\hat{G}^{(s)}) = \hat{\mathbf{g}}^{(s)}(\hat{F}^{(s)}) \tag{27}$$
$$g(q,p) f(q,p) = Tr\{[\hat{\mathbf{g}}^{(s)}(\hat{\mathbf{F}}^{(s)})]\hat{\Delta}_{qp}(-s)\} = Tr\{[\hat{\mathbf{f}}^{(s)}(\hat{\mathbf{G}}^{(s)})]\hat{\Delta}_{qp}(-s)\} \tag{28}$$

These two relations explicitly answer the first question stated in the Introduction. Under the WWGM association corresponding to an arbitrary s-ordering, to the product of two c-number functions there corresponds an operator which results by action of the Liouvillian superoperator form of one on the other. More formally, in accordance with (19), we obtain

$$\mathcal{M}_s[g(q,p) f(q,p)] = \hat{\mathbf{g}}^{(s)}(\hat{F}^{(s)}) = \hat{\mathbf{f}}^{(s)}(\hat{\mathbf{G}}^{(s)}) \tag{29}$$

As an example, by making use of Eq. (11), we have

$$\mathcal{M}_s(q^{n_1} p^{m_1} q^{n_2} p^{m_2}) = \hat{\mathbf{O}}^{(s)}_{n_1 m_1}(\hat{t}^{(s)}_{n_2 m_2}) = \hat{\mathbf{O}}^{(s)}_{n_2 m_2}(\hat{t}^{(s)}_{n_1 m_1}) = \hat{t}^{(s)}_{n_1+n_2, m_1+m_2}$$

Note that the result is, in general, different from the noncommutative product $\hat{t}^{(s)}_{n_2 m_2}\hat{t}^{(s)}_{n_1 m_1}$ or from $\hat{t}^{(s)}_{n_1 m_1}\hat{t}^{(s)}_{n_2 m_2}$.

4. DERIVATION OF THE NEW BRACKET

Taking the derivatives of the second relation in (19) with respect to q (and p) and then making use of Eq. (23), we have

$$\partial_p f(q,p) = Tr[\hat{F}^{(s)}\, \partial_p(\hat{\Delta}_{qp}(-s))]$$
$$= -\frac{i}{\hbar}\, Tr[(\mathbf{ad}_{\mathbf{q}} \hat{F}^{(s)})\hat{\Delta}_{qp}(-s)] \tag{30}$$
$$\partial_q g(q,p) = Tr[\hat{G}^{(s)} \partial_q(\hat{\Delta}_{qp}(-s))]$$
$$= \frac{i}{\hbar}\, Tr[(\mathbf{ad}_{\mathbf{p}} \hat{G}^{(s)})\hat{\Delta}_{qp}(-s)] \tag{31}$$

where $\mathbf{ad}_{\hat{\mathbf{A}}}$ denotes the adjoint action: $\mathbf{ad}_{\hat{\mathbf{A}}}\hat{B} = [\hat{A}, \hat{B}]$. These relations show that, if $\mathcal{M}_s(f) = \hat{F}^{(s)}$, then $\mathcal{M}_s(\partial_p f) = -(i/\hbar)\mathbf{ad}_{\hat{\mathbf{q}}}\hat{F}^{(s)}$ and $\mathcal{M}_s(\partial_q f) = (i/\hbar)\mathbf{ad}_{\hat{\mathbf{p}}}\hat{F}^{(s)}$. Now by multiplying both sides of Eq. (30) by $\partial_q g$ and making use of (26) and then of (12), we have

$$\partial_p f\, \partial_q g = -\frac{i}{\hbar}\, Tr[(\mathbf{ad}_{\mathbf{q}} \hat{F}^{(s)})\hat{\mathbf{g}}^{(-s)}_{\mathbf{q}}(\hat{\Delta}_{qp}\ (-s))]$$
$$= -\frac{i}{\hbar}\, Tr[\hat{\Delta}_{qp}(-s)\hat{\mathbf{g}}^{(s)}_{\mathbf{q}}(\mathbf{ad}_{\mathbf{q}} \hat{F}^{(s)})] \tag{32}$$

By reversing the order of calculations, these relations can be rewritten as

$$\partial_p f\, \partial_q g = \frac{i}{\hbar}\, Tr[\hat{\Delta}_{qp}(-s)\hat{\mathbf{f}}^{(s)}_{\mathbf{p}}(\mathbf{ad}_{\mathbf{p}}\hat{G}^{(s)})] \tag{33}$$

where $\hat{\mathbf{h}}^{(s)}_{\mathbf{x}}$ stands for the superoperator associated to the $\partial_x h$. In a similar way we have

$$\partial_q f\, \partial_p g = \frac{i}{\hbar}\, Tr[\hat{\mathbf{g}}^{(s)}_{\mathbf{p}}(\mathbf{ad}_{\mathbf{p}}\hat{F}^{(s)})\hat{\Delta}_{qp}(-s)] \tag{34}$$

$$= -\frac{i}{\hbar}\, Tr[\hat{\mathbf{f}}^{(s)}_{\mathbf{q}}(\mathbf{ad}_{\mathbf{q}}\hat{G}^{(s)})\hat{\Delta}_{qp}(-s)] \tag{35}$$

Thus, by combining Eqs. (32) and (33) with (34) and (35), we arrive at four differently looking, but equivalent expressions for the PB of two functions:

$$\{f, g\}_{\mathrm{PB}} = -\frac{i}{\hbar}\, Tr\{[\hat{\mathbf{g}}^{(s)}_{\mathbf{q}}(\mathbf{ad}_{\hat{\mathbf{q}}}\hat{F}^{(s)}) + \hat{\mathbf{g}}^{(s)}_{\mathbf{p}}(\mathbf{ad}_{\hat{\mathbf{p}}}\hat{F}^{(s)})]\hat{\Delta}_{qp}(-s)\} \tag{36}$$

$$= -\frac{i}{\hbar}\, Tr\{[\hat{\mathbf{g}}^{(s)}_{\mathbf{q}}(\mathbf{ad}_{\hat{\mathbf{q}}}\hat{F}^{(s)}) - \hat{\mathbf{f}}^{(s)}_{\mathbf{q}}(\mathbf{ad}_{\hat{\mathbf{q}}}\hat{G}^{(s)})]\hat{\Delta}_{qp}(-s)\} \tag{37}$$

$$= \frac{i}{\hbar}\, Tr\{[\hat{\mathbf{f}}^{(s)}_{\mathbf{p}}(\mathbf{ad}_{\hat{\mathbf{p}}}\hat{G}^{(s)}) - \hat{\mathbf{g}}^{(s)}_{\mathbf{p}}(\mathbf{ad}_{\hat{\mathbf{p}}}\hat{F}^{(s)})]\hat{\Delta}_{qp}(-s)\} \tag{38}$$

$$= \frac{i}{\hbar}\, Tr\{[\hat{\mathbf{f}}^{(s)}_{\mathbf{q}}(\mathbf{ad}_{\hat{\mathbf{q}}}\hat{G}^{(scp}) + \hat{\mathbf{f}}^{(s)}_{\mathbf{p}}(\mathbf{ad}_{\hat{\mathbf{p}}}\hat{G}^{(s)})]\hat{\Delta}_{qp}(-s)\} \tag{39}$$

These relations enable us to define a new bracket in $\mathcal{F}(\mathcal{H})$, which we denote by $[\ ,\ ]^{(s)}_{\mathrm{PMB}}$, and call the Poisson–Moyal bracket (PMB). It is defined as the image of the PB under the WWGM association:

$$\mathcal{M}_s(\{f, g\}_{\mathrm{PB}}) = [\hat{F}, \hat{G}]^{(s)}_{\mathrm{PMB}} \tag{40}$$

and is explicitly given by the following four equivalent expressions:

$$[\hat{F}, \hat{G}]^{(s)}_{\mathrm{PMB}} = -\frac{i}{\hbar}\, [\hat{\mathbf{g}}^{(\mathbf{s})}_{\mathbf{q}}(\mathbf{ad}_{\hat{\mathbf{q}}}\hat{F}^{(s)}) + \hat{\mathbf{g}}^{(\mathbf{s})}_{\mathbf{p}}(\mathbf{ad}_{\hat{\mathbf{p}}}\hat{F}^{(s)})] \tag{41}$$

$$= -\frac{i}{\hbar}\, [\hat{\mathbf{g}}^{(\mathbf{s})}_{\mathbf{q}}(\mathbf{ad}_{\hat{\mathbf{q}}}\hat{F}^{(s)}) - \hat{\mathbf{f}}^{(\mathbf{s})}_{\mathbf{q}}(\mathbf{ad}_{\hat{\mathbf{q}}}\hat{G}^{(s)})] \tag{42}$$

$$= \frac{i}{\hbar}\, [\hat{\mathbf{f}}^{(\mathbf{s})}_{\mathbf{p}}(\mathbf{ad}_{\hat{\mathbf{p}}}\hat{G}^{(s)}) - \hat{\mathbf{g}}^{(\mathbf{s})}_{\mathbf{p}}(\mathbf{ad}_{\hat{\mathbf{p}}}\hat{F}^{(s)})] \tag{43}$$

$$= \frac{i}{\hbar}\, [\hat{\mathbf{f}}^{(\mathbf{s})}_{\mathbf{q}}(\mathbf{ad}_{\hat{\mathbf{q}}}\hat{G}^{(s)}) + \hat{\mathbf{f}}^{(\mathbf{s})}_{\mathbf{p}}(\mathbf{ad}_{\hat{\mathbf{p}}}\hat{G}^{(s)})]. \tag{44}$$

Obviously, since the WWGM association is linear, and the PB is a Lie bracket, i.e., bilinear, antisymmetric, and obeying the Jacobi identity, so is this new PMB. The four seemingly different, but equivalent expressions of this new bracket correspond to trivially equivalent rearrangement of the terms on the right-hand side of (17). In contrast, the equivalence of Eqs. (41)–(44) is not so trivial.

In the case of many degrees of freedom, on the right-hand sides of Eqs. (41)–(44) $\hat{q}$ and $\hat{p}$ are to be labeled with an index and summed over. For instance, the second and third relations are

$$[\hat{F}, \hat{G}]^{(\mathbf{s})}_{\mathrm{PMB}} = -\frac{i}{\hbar} \sum_i [\hat{\mathbf{g}}^{(\mathbf{s})}_{\mathbf{q}_i}(\mathbf{ad}_{\hat{\mathbf{q}}_i}\hat{F}^{(s)}) - \hat{\mathbf{f}}^{(\mathbf{s})}_{\mathbf{q}_i}(\mathbf{ad}_{\hat{\mathbf{q}}_i}\hat{G}^{(s)})] \tag{45}$$

$$= -\frac{i}{\hbar} \sum_i [\hat{\mathbf{f}}^{(\mathbf{s})}_{\mathbf{p}_i}(\mathbf{ad}_{\hat{\mathbf{p}}_i}\hat{G}^{(s)}) - \hat{\mathbf{g}}^{(\mathbf{s})}_{\mathbf{p}_i}(\mathbf{ad}_{\hat{\mathbf{p}}_i}\hat{F}^{(s)})] \tag{46}$$

5. APPLICATIONS

As a general first application, we take $f(q, p) = q^n p^m$ and $g(q, p) = q^k p^l$, n, m, k, l integers. These kinds of monomials form a basis for the so called w_∞-algebra with respect to PB:

$$\{q^n p^m, q^k p^l\}_{\mathrm{PB}} = (mk - nl) q^{n+k-1} p^{m+l-1} \tag{47}$$

and the W_∞ algebra with respect to s-MB [9],

$$\{q^n p^m, q^k p^l\}^{(-s)}_{\mathrm{MB}} = \sum_{j=0}^{j_{\max}} \frac{i^j}{j!} \left[{\sum_{r=0}^{j}}' \binom{j}{r} f_{srj} a_{nmkl,rj} \right] q^{n+k-j} p^{m+l-j} \tag{48}$$

Here the prime on the second summation indicates that the maximum value that r may take is $r_{\max} = (m, k)$ (i.e., the smaller of the integers m and k) and

$$j_{\max} = (n + r_{\max}, l + r_{\max}),$$

$$a_{nmkl,rj} = \frac{n!m!k!l!}{(n + r - j)!\,(m - r)!\,(k - r)!\,(l + r - j)!} \tag{49}$$

The restrictions imposed on the summations also follow from the expression of $a_{nmkl,rj}$. In Eq. (44)

$$f_{srj} = (s^-)^r (-s^+)^{j-r} - (s^-)^{j-r} (-s^+)^r$$

is the only factor depending on the chosen rule of ordering, where $s^\pm = \hbar(1 \pm s)/2$. The w_∞-algebra is the algebra of canonical diffeomorphisms of a phase space that is topologically equivalent to $\mathbf{R}^2$, or, since the area element and symplectic form coincide in two dimensions, as the algebra of area-preserving diffeomorphisms $Diff_A\ \mathbf{R}^2$ [13]. The above W_∞-algebra is the quantum (or $\hbar$) deformation of this classical w_∞. More explicitly, one can easily show that

$$\lim_{\hbar\to 0} (i\hbar)^{-1}\{\ ,\ \}^{(-s)}_{\rm MB} = \{\ ,\ \}_{\rm PB} \tag{50}$$

Now with respect to our new bracket we will obtain an algebra isomorphic to the w_∞-algebra. This will be denoted by $\mathcal{F}_{\rm PMB}$. From (19) we obtain

$$\hat{F}^{(s)} = \hat{t}^{(s)}_{nm}, \qquad \hat{G}^{(s)} = \hat{t}^{(s)}_{kl} \tag{51}$$

and by making use of (7),

$$\mathbf{ad}_{\hat{\mathbf{q}}}\hat{F}^{(s)} = i\hbar m\hat{t}^{(s)}_{n,m-1}, \qquad \mathbf{ad}_{\hat{\mathbf{p}}}\hat{F}^{(s)} = -i\hbar n t^{(s)}_{n-1,m} \tag{52}$$

$$\mathbf{ad}_{\hat{\mathbf{q}}}\hat{G}^{(s)} = i\hbar l\hat{t}^{(s)}_{k,l-1}, \qquad \mathbf{ad}_{\hat{\mathbf{p}}}\hat{G}^{(s)} = -i\hbar k\hat{t}^{(s)}_{k-1,l} \tag{53}$$

The corresponding superoperators are as follows:

$$\begin{aligned}
\hat{\mathbf{f}}^{(s)}_q &= n2^{-(n+m-1)}\hat{\mathbf{T}}^{n-1}_{[\hat{q}](s)}\hat{\mathbf{T}}^{m}_{[\hat{p}](-s)} = n\hat{\mathbf{O}}^{(s)}_{n-1,m}\\
\hat{\mathbf{f}}^{(s)}_p &= m\hat{\mathbf{O}}^{(s)}_{n,m-1}\\
\hat{\mathbf{g}}^{(s)}_q &= k\hat{\mathbf{O}}^{(s)}_{k-1,l}, \qquad \hat{\mathbf{g}}^{(s)}_p = l\hat{\mathbf{O}}^{(s)}_{k,l-1}
\end{aligned} \tag{54}$$

Substituting these relations into any of Eqs. (41)–(44) and using identities such as [see Eq. (11)]

$$\hat{\mathbf{O}}^{(s)}_{k-1,l}(\hat{t}^{(s)}_{n,m-1}) = \hat{\mathbf{O}}^{(s)}_{k,l-1}(\hat{t}^{(s)}_{n-1,m}) = \hat{t}^{(s)}_{n+k-1,m+l-1}$$

we obtain

$$[\hat{F}, \hat{G}]^{(s)}_{\rm PMB} = (mk - nl)\hat{t}^{(s)}_{n+k-1,m+l-1} \tag{55}$$

Thus, by comparing with (47), we see that $\mathcal{F}_{\rm PMB}$ is isomorphic to the w_∞-algebra. Because of (50), or as can be directly verified, we have

$$-\lim_{\hbar\to 0} (i\hbar)^{-1}[\ ,\ ] = [\ ,\ ]_{\rm PMB} \tag{56}$$

provided that the same ordering convention is used on both sides.

There are some remarkable particular cases of this general application that deserve to be mentioned. W_∞-algebra has abelian and finite- or infinite-dimensional non-Abelian subalgebras for which the structure constants are proportional to the first power of $i\hbar$ [9]. These are generated by $\hat{t}^{(s)}_{nm}$ such that (i) $n = 0$, (ii) $m = 0$, (iii) $n = m$ (Cartan subalgebra) [14], (iv) $n + m \leq 1$ (HW-algebra), (v) $n + m = 2$ [symplectic algebra $sp(2)$], (vi) $n + m \leq 2$ [inhomogeneous symplectic algebra $isp(2)$], (vii) $m = 1$, and (viii) $n = 1$. The first three are infinite-dimensional Abelian subalgebras and the last two are isomorphic copies of the centerless Virasoro algebra [13]. For all these subalgebras, Eq. (56) is of the form $-(i\hbar)^{-1}[\ ,\ ] = [\ ,\ ]_{\text{PMB}}$.

As a second general application, we will carry the classical Hamiltonian equations of motion

$$\dot{q} = -\{q, H\}_{\text{PB}} = \partial_p H, \qquad \dot{p} = -\{p, H\}_{\text{PB}} = -\partial_q H \tag{57}$$

to $\mathcal{F}(\mathcal{H})$. Here $H \equiv H(q, p)$ is the classical Hamiltonian and t is the time parameter $\dot{a} \equiv da/dt$. Now applying $\mathcal{M}_s$ to both sides of Eq. (57), we obtain

$$\dot{\hat{q}} = \frac{1}{i\hbar}[\hat{q}, \hat{H}^{(s)}], \qquad \dot{\hat{p}} = \frac{1}{i\hbar}[\hat{p}, \hat{H}^{(s)}] \tag{58}$$

Here $\hat{H}^{(s)} = \mathcal{M}_s(H)$, and we used the fact that

$$[\hat{q}, \hat{H}]^{(s)}_{\text{PMB}} = (i/\hbar)[\hat{q}, \hat{H}^{(s)}], \qquad [\hat{q}, \hat{H}]^{(s)}_{\text{PMB}} = (i/\hbar)[\hat{p}, \hat{H}^{(s)}] \tag{59}$$

Notice that $\hat{H}^{(s)}$ is Hermitian only for pure imaginary values of s, that is, $(\hat{H}^{(s)})^{\dagger} = \hat{H}^{(-\bar{s})}$. In particular, when $H = (p^2/2m) + V(q)$, Eqs. (58) are of the form (Ehrenfest's theorem)

$$\dot{\hat{q}} = \frac{\hat{p}}{m}, \qquad \dot{\hat{p}} = -\frac{i}{\hbar}[\hat{p}, \hat{V}(\hat{q})] \tag{60}$$

Note that these are independent of s.

We call Eqs. (58) the operator form of the Hamilton equations. Assume that the operators belong to the Heisenberg picture; these equations are identical to Heisenberg-picture equations of motion that can be obtained from

$$\frac{d\hat{A}_{\text{H}}}{dt} = \frac{\partial \hat{A}_{\text{H}}}{\partial t} + \frac{1}{i\hbar}[\hat{A}_{\text{H}}, \hat{H}] \tag{61}$$

by taking $\hat{A}_{\text{H}} = \hat{q}, \hat{p}$ and $\hat{H} = \hat{H}^{(s)}$. Here the subscript H refers to the Heisenberg picture, in which the state vectors are time independent and the dynamical variables are time dependent. Note that the first term on the right-side of Eq. (61) is defined as follows [15]:

$$\frac{\partial \hat{A}_{\text{H}}}{\partial t} \equiv \left(\frac{\partial \hat{A}}{\partial t}\right)_{\text{H}} = \hat{U}\,\frac{\partial \hat{A}_{\text{S}}}{\partial t}\,\hat{U}^{\dagger} \tag{62}$$

where $\hat{U} = \exp(it\hat{H}/\hbar)$ is the evolution operator and the subscript S refers to the Schrödinger picture, in which the state vectors are time dependent and dynamical variables are time independent (except for a possible explicit time dependence, which is not the case for the position and momentum operators in the Schrödinger picture).

As a result, as far as the dynamics of $\hat{q}$ and $\hat{p}$ are concerned, the WWGM association directly maps the classical Hamilton equations of motion onto the Heisenberg-picture equations of motion for general Hamiltonian $\hat{H}^{(s)} = \mathcal{M}_s(H)$. Attaining the Schrödinger equation (see the next section) for the time evolution of state vectors is straightforward by making use of the evolution operator $\hat{U}$ in the case of $\bar{s} = -s$. Despite this application, the fact that the image of classical mechanics under the WWGM association is not identical to the Heisenberg-picture formulation of conventional quantum mechanics is made apparent in the next application.

Finally, we consider the equation describing the time evolution of a phase-space function $f \equiv f(q, p; t)$,

$$\dot{f} = \partial_t f + \{H, f\}_{\mathrm{PB}} \tag{63}$$

associated with a system described by H. The corresponding equation in $\mathcal{F}(\mathcal{H})$ is

$$\dot{\hat{F}}^{(s)} = \partial_t \hat{F}^{(s)} + [\hat{H}, \hat{F}]^{(s)}_{\mathrm{PMB}} \tag{64}$$

where $\hat{F}^{(s)} = \mathcal{M}_s(f)$. In particular, for $H = (p^2/2m) + V(q)$, $f = f(q)$, and $g = g(p)$ the equations are

$$\dot{\hat{F}}^{(\mathbf{s})} = \frac{1}{m}\,\hat{\mathbf{f}}^{(s)}_{\mathbf{q}}(\hat{p}), \qquad \dot{\hat{G}}^{(s)} = -\frac{i}{\hbar}\,\hat{\mathbf{g}}^{(\mathbf{s})}_{\mathbf{p}}[\hat{p}, \hat{V}(\hat{q})] \tag{65}$$

Note that the operator form of the Hamilton equations is a particular case of these last equations. The distinction between the Heisenberg-picture formulation of quantum mechanics and the image of the Hamilton formulation of classical mechanics is made manifest in (64) by the appearance of PMB instead of [,].

6. CONCLUSION AND DISCUSSION

The conventional way of finding a quantum system that reduces to a specified classical system in the classical limit is to write the classical Hamilton equations in terms of the PB and then replace the PB with commutator brackets in accordance with $\{f, g\}_{\mathrm{PB}} \to (i\hbar)^{-1}[\hat{F}, \hat{G}]$. Although this construction suffers from the difficulties stated in the Introduction, the association $\{f, g\}_{\mathrm{PB}} \to [\hat{F}, \hat{G}]^{(s)}_{\mathrm{PMB}}$ is free of them. Note that $-(i\hbar)^{-1}[\ ,\ ]^{(s)}_{\mathrm{PMB}}$ reduces to the commutator bracket when one of the entries is $\hat{q}$ or $\hat{p}$. A bit more generally, when $\hat{F} = a\hat{q} + b\hat{p} + c\hat{I}$, a general element of the HW-algebra, then $-[\hat{F}, \hat{H}]^{(s)}_{\mathrm{PMB}} = (i\hbar)^{-1}[\hat{F}, \hat{H}^{(s)}]$. Thus, for a time-independent Hamiltonian, as in the solutions of Eqs. (58), the time evolution of the basic observables is given by $\hat{q}(t) = \hat{U}(t, s)\hat{q}(0)\hat{U}(-t, s)$, $\hat{p}(t) = \hat{U}(t, s)\hat{p}(0)\hat{U}(-t, s)$. Here $\hat{q}(0)$ and $\hat{p}(0)$ are time-independent position and momentum operators and $\hat{U}(t, s) = \exp(it\hat{H}^{(s)}/\hbar)$. By noting that $\hat{U}(t, s)$ is unitary only when $\bar{s} = -s$, we see that these are the same as in the Heisenberg picture if $\hat{q}(0)$ and $\hat{p}(0)$ are considered to be in the Schrödinger picture and if $\bar{s} = -s$. In that case, by assuming time-independent state vector $|\psi(0)\rangle \in \mathcal{H}$ in the Heisenberg picture such that $\langle\psi(0)|\hat{p}(t)|\psi(0)\rangle = \langle\psi(t)|\hat{p}(0)|\psi(t)\rangle$, we obtain a time-dependent state vector $|\psi(t)\rangle = \hat{U}(-t, s)|\psi(0)\rangle$ obeying the dynamics $i\hbar\partial_t|\psi(t)\rangle = \hat{H}^{(s)}|\psi(t)\rangle$, i.e., the Schrödinger equation.

On the other hand, while the time evolution of a general observable (not explicitly time dependent) is given in the Heisenberg picture by $\hat{F}_{\mathrm{H}}(t) = \hat{U}(t)\hat{F}_{\mathrm{H}}(0)\hat{U}(t)^{\dagger}$, it is not so in the association scheme PB $\to$ PMB. Instead, if we define $(\mathbf{ad}_{\hat{\mathbf{A}}})^{(s)}_{\mathrm{PM}}$ by $(\mathbf{ad}_{\hat{\mathbf{A}}})^{(s)}_{\mathrm{PM}}\hat{B} = [\hat{A}, \hat{B}]^{(s)}_{\mathrm{PMB}}$, then the time evolution governed by Eq. (64) can be written as $\hat{F}^{(s)}(t) = [\exp - t(\mathbf{ad}_{\hat{\mathbf{H}}})^{(s)}_{\mathrm{PM}}]\hat{F}^{(s)}(0)$, which, because of (50) or (56), is the limiting case of the above relation.

In the sense described above, the conventional canonical quantization itself can be thought of as an $\hbar$ deformation of the quantization by PMB. This fact is made manifest by the following diagrams:

(i) Hierachy of products:

$$\begin{array}{ccc} f.g = g.f & \Leftarrow \text{WWGM association} \Rightarrow & \hat{F} \diamondsuit \hat{G} = \hat{G} \diamondsuit \hat{F} \\ \text{def.} \downarrow\uparrow \text{cont.} & & \text{def.} \downarrow\uparrow \text{cont.} \\ f * g \neq g * f & \Leftarrow \text{WWGM association} \Rightarrow & \hat{F}\hat{G} \neq \hat{G}\hat{F} \end{array}$$

(ii) Hierachy of brackets:

$$\begin{array}{ccc} \{.,.\}_{\mathrm{PB}} & \Leftarrow \text{WWGM association} \Rightarrow & [.,.]_{\mathrm{PMB}} \\ \text{def.} \downarrow\uparrow \text{cont.} & & \text{def.} \downarrow\uparrow \text{cont.} \\ \{.,.\}_{\mathrm{MB}} & \Leftarrow \text{WWGM association} \Rightarrow & [.,.] \end{array}$$

These two diagrams schematically summarize the main points of this paper and exhibit the hierarchies of the products and brackets involved. Here cont. and def. stand for contraction and deformation, respectively, and, for the sake of simplicity, the commutative operator product derived in Eq. (29) is shown here by $\hat{\mathbf{f}}(\hat{G}) \equiv \hat{F} \diamondsuit \hat{G}$. Note that the $\diamondsuit$ product (like the $*$ product and the MB) depends on the ordering parameter s, but this is not shown in the diagrams, for the same reason. In summary, what we have done here may be considered "contraction quantization," or the formulation of classical mechanics in the ring of operators. We study this direction in a forthcoming paper.

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